ASYMPTOTIC BEHAVIOR OF SOME HANKEL-TOEPLITZ DETERMINANTS

George A. Baker, Jr.,* D. Bessis† and P. Moussa†

*Theoretical Division, Los Alamos National Laboratory
University of California, Los
Alamos, New Mexico 87545 USA
and
†Service de Physique Théorique,§Centre
d'Études Nucléaires de Saclay
91191 Gif-sur-Yvette, Cedex France

Abstract. Motivated by the recent increase in interest in Toeplitz determinants of large order, whose elements are moments with respect to measures, by their connection with the theory of quantum gravity we have given exact values of the determinants for several large classes of measures. These classes are related to the continued fraction of Gauss and its q-extension. We have proven that the q-extension, which is related to the theory of quantum groups, is unique. In addition we have extended the work of Szegő from measures of finite support to a wide class of those with infinite support.

I. Introduction and Summary

Ambjørn *et al.* [1], David [2] and Kazakov *et al.* [3] proposed that the integral over the internal geometry of a surface could be discretized as the integral over randomly triangulated surfaces. When this is done, then the partition functions of certain models of two-dimensional quantum gravity can be expressed in terms of the free energy of the associated Hermitian matrix models. These models can frequently be solved asymptotically in the limit of a large number N of degrees of freedom, as was pointed out by Brézin *et al.* [4] and an asymptotic expansion

§Laboratoire de la Direction des Sciences de la Matière du Commissariat á l'Energie Atomique

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

in N^{-1} or topologic expansion was given by Bessis *el al.* [5]. Resting on this foundation, recently the continuum limit of a sum over the topologies of two-dimensional surfaces was defined for certain model of two dimensional gravity and it was reasoned that they correspond to two dimensional quantum gravity coupled to minimal conformal matter (Douglas and Shenkar [6], Brézin and Kazakov [7], and Gross and Migdal [8]). See Zinn-Justin [9] for a recent review of this subject.

The mathematical connection (following Bessis *et al.* [5]) between this formulation and the Toeplitz determinant problem is as follows: We begin with the quantity,

$$Z_N(g) = \int_{-\infty}^{\infty}\cdots\int dM\, e^{-\mathrm{tr}(V(M))}, \tag{1.1}$$

where M is an element of the N^2-dimensional real vector space of $N \times N$ Hermitian matrices, and

$$\mathrm{tr}(V(M)) = \frac{1}{2}\mathrm{tr}(M^2) + \sum_{p\geq 2}^{v} \frac{g_p}{N^{p-1}}\mathrm{tr}\ (M^{2p}), \tag{1.2}$$

where, depending on the case, v is either finite (V a polynomial) or infinite. This real vector space carries a representation of the group of unitary $N \times N$ matrices U such that,

$$M \overset{U}{\mapsto} UMU^{-1}. \tag{1.3}$$

We can restrict $\det U = 1$ as any phase factor cancels out of (1.3). The Lebesgue measure,

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1\leq i<j}^{N} d\left(\mathrm{Re}\ M_{ij}\right) d\left(\mathrm{Im}\ M_{ij}\right), \tag{1.4}$$

is invariant under the unitary transformations (1.3). Thus (Mehta [10]) using the special case where M is the diagonal matrix Λ with diagonal entries λ_i, we can rewrite (1.1) as

$$Z_N(g) = \Omega_N(2\pi)^{-N}\int_{-\infty}^{\infty}\cdots\int \prod_{i=1}^{N} d\lambda_i\, \Delta^2(\lambda) \exp\left(-\sum_{i=1}^{N} V(\lambda_i)\right), \tag{1.5}$$

where,

$$V(\lambda) = \frac{1}{2}\lambda^2 + \sum_{p\geq 2} \bar{g}_p\lambda^{2p}, \quad \bar{g}_p = \frac{g_p}{N^{p-1}}. \tag{1.6}$$

The quantity Ω_N is related to the volume of the unitary group and is,

$$\Omega_N = \frac{2^N \pi^{\frac{1}{2}N(N+1)}}{\prod_{p=1}^{N} p!}. \tag{1.7}$$

Finally, the quantity $\Delta(\lambda)$ is an $N \times N$ Van der Monde determinant,

$$\Delta(\lambda) = \prod_{1 \le i < j}^{N} (\lambda_i - \lambda_j) = \det \parallel \lambda_i^{j-1} \parallel . \tag{1.8}$$

If we take,

$$d\mu(\lambda) = e^{-V(\lambda)}, \tag{1.9}$$

then, following Bessis [11] we can re-write the form (1.5) of (1.1) as proportional to

$$\begin{aligned}
\bar{Z}_N(g) &= \int_{-\infty}^{\infty} \cdots \int \Delta^2(\lambda) \prod_{i=1}^{N} d\mu(\lambda_i), \\
&= \int_{-\infty}^{\infty} \cdots \int \det \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^{N-1} \end{vmatrix}^2 \prod_{i=1}^{N} d\mu(\lambda_i).
\end{aligned} \tag{1.10}$$

If we expand one of the determinants and recognize that every term we get is just a permutation of every other term, then

$$\begin{aligned}
&= N! \int_{-\infty}^{\infty} \cdots \int \lambda_1^0 \lambda_2^1 \cdots \lambda_N^{N-1} \det \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \cdots & \lambda_N^{N-1} \end{vmatrix} \prod_{i=1}^{N} d\mu(\lambda_i) \\
&= N! \int_{-\infty}^{\infty} \cdots \int \det \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{N-1} \\ \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_N^{N-1} & \lambda_N^N & \cdots & \lambda_N^{2N-2} \end{vmatrix} \prod_{i=1}^{N} d\mu(\lambda_i), \\
&= N! \det \parallel a_{i+j-2} \parallel \equiv N! D(N),
\end{aligned} \tag{1.11}$$

where

$$a_j = \int_{-\infty}^{\infty} \lambda^j \, d\mu(\lambda). \tag{1.12}$$

which implies that $Z_N(G)$ is proportional to an $N \times N$ Toeplitz determinant of the moments of a distribution!

There are other cases of interest where the power of $\Delta(\lambda)$ in (1.5) differs from two. They correspond to the powers 1 and 4. The case of $\Delta(\lambda)$ arises from considering (1.1) for the case of real symmetric $N \times N$ matrices, which is invariant under the orthogonal group, and the case $\Delta^4(\lambda)$ arises from considering (1.1) as quaternion self-dual matrices or all self-dual unitary matrices, which are invariant under the symplectic group (Mehta and Mahoux [12]). The occurrence of these cases is related to the existence of only three associative division algebras, the real numbers, the complex numbers and the quaternions. In addition there is some connection with the study of Coulomb gases, where the λ_i of (1.5) are thought of as positions and the exponent of $\Delta(\lambda)$ is related to the interaction strength (Nienhuis [13]).

Another important, relevant case of related study is the work of Szegő [14, 15] on the asymptotic behavior for large N of the forms (1.11) subject to the restriction (1.12), however with a *finite* length interval as the support of the measure. The present interest goes beyond the interests of Szegő in two ways, first we wish to consider measures of infinite support and second we would like to find asymptotic expansions in N^{-1} for these properties.

In the second section of this paper, we given some classically known background material which shows the relation of the Toeplitz determinants to the theory of orthogonal polynomials. We give the results of Szegő [14] on the asymptotic behavior of the Toeplitz determinants and collect some of the exact results which are known for this behavior from the normalization integrals of classical families of orthogonal polynomials. Finally we give Selberg's integral and the limiting case for an infinite range of integration.

In the third section we discuss the results of the continued fraction of Gauss and show how the exact behavior of a wide class of Toeplitz determinants can be computed from known results.

In the fourth section we introduce the q-extension of the continued fraction of Gauss, that is for ratios of the "basic" hypergeometric functions. These results allow us to compute the exact behavior of an even wider class of Toeplitz determinants, which permits the asymptotic expansions in $1/N$ with N the order of the determinant to be computed to any desired degree.

In the fifth section we consider whether any further extensions of the results of sections three and four of the same type are possible. We replace the q-mapping with a general mapping. We find that the requirement that the resulting generalized hypergeometric functions have contiguous relations, implies that there are no further generalizations of this character beyond the "basic" hypergeometric functions. Since the q-extension is closely related to the theory of quantum groUps, we speculate that theSe ReSulTS have implications in that theory as well.

In the sixth section, we use the saddle-point method of Brézin *et al.* [4] to compute the asymptotic behavior to leading orders of Toeplitz determinants corresponding to an extensive class of measures with only a finite range of support, extending the results of Szegő [14] from measures of finite support to a wide class of those with infinite support.

In the seventh section, we consider some measures of the type that occur in quantum gravity problems which depend on the order of the Toeplitz determinants.

II. Some Background

In this section we remind the reader of some relevant known results. One of the key objects of study of this paper is the asymptotic behavior of

$$D(N) = \det |a_{i+j-2}|_{1\leq i,j\leq N}\,, \tag{2.1}$$

as defined by (1.11-12). This object arises in the theory of orthogonal polynomials to which a great deal of study has been given. Suppose we are given a measure $d\phi(u)\geq 0$ supported over some portion of the real u-axis. We can then define a family of monic, orthogonal polynomials $\psi_n(u),\ n=0,1,\ldots$ such that

$$\int \psi_m(u)\psi_n(u)\, d\phi(u) = 0, \quad n\neq m, \tag{2.2}$$

$$\psi_m(u) = \sum_{\nu=0}^{m} \psi_{m,\nu}u^\nu, \quad \psi_{m,m}=1. \tag{2.3}$$

We mean by a monic polynomial one for which the coefficient of its argument to the highest poser os imotu. as om (2.3). By the direct solution by Cramer's rule of the linear algebraic equations implied by (2.2) for the coefficients $\psi_{m,\nu}$ and substitution in the normalization integral, we obtain the relation between the orthogonal polynomials and the Toeplitz determinants,

$$N(m) \equiv \int \psi_m^2(u)\, d\phi(u) = \int u^m\psi_m(u)\, d\phi = \frac{D(m+1)}{D(m)}, \tag{2.4}$$

where the first equality follows by the orthogonality properties. By definition $D(0)=1$, we may use (2.4) to generate recursively the $D(m)$

from the orthogonal polynomial normalization integrals, $N(m)$. It is interesting to note that $N(m)$ is a minimum with resprect to variations in $\psi_{m,\nu}$ subject to the condition $\psi_{m,m} = 1$.

The results of Szegő [14] on the asymptotic properties of the normalization constant $N(m)$ are briefly as follows. First, let us take $d\phi(u) = \exp(-V(u))\,du$ in consentience with the notation of Sec. 1, but over a finite range $a \le u \le b$. Here $V(u)$ is not assumed to be a polynomial. His results are:

$$\lim_{m\to\infty}\left[\left(\frac{4}{b-a}\right)^{2m+1} N(m)\right] = 2\pi \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi} V\left(\frac{1}{2}(b-a)\cos\theta + \frac{1}{2}(a+b)\right)\,d\theta\right\} \equiv \mathcal{V}. \quad (2.5)$$

For large m the normalization constant behaves geometrically. Consequently, we derive easily that,

$$\sqrt[m]{D(m)} \asymp \mathcal{V}\left(\frac{b-a}{4}\right)^m. \quad (2.6)$$

We list now a few of the known results, taken for the tables of Morse and Feshbach [16] for the constant $N(m)$. First, the Gegenbauer polynomials, $T_m^\beta(x)$, are orthogonal over the interval -1 to $+1$ with the measure $d\phi(u) = (1-u^2)^\beta\,du$. The special case $\beta = 0$ gives the Legendre polynomials, the case $\beta = n$ gives the associated Legendre functions, $P_m^n(x) = (1-x^2)^{n/2}T_m^n(x)$, and the cases $\beta = \pm\frac{1}{2}$ are the Tschebyscheff polynomials. The normalization integral of (2.4) is for the (monic) Gegenbauer polynomials,

$$N(m) = \frac{2^{2m+2\beta+1}\Gamma(m+2\beta+1)\Gamma(m+1)\Gamma(m+\beta+1)^2}{(2m+2\beta+1)\Gamma(2m+2\beta+1)^2}. \quad (2.7)$$

Next the Jacobi polynomials are orthogonal over the interval, 0 to $+1$, with the weight function $d\phi(u) = u^{c-1}(1-u)^{a-c}\,du$. There are a number of special cases. If $a = 2\beta+1$ and $c = \beta+1$ the Jacobi polynomials are proportional to the Gegenbauer polynomials $T_m^\beta(1-2x)$. The normalization constants for the (monic) Jacobi polynomials is

$$N(m) = \frac{\Gamma(m+1)\Gamma(m+a)\Gamma(m+c)\Gamma(m+a-c+1)}{(a+2m)\Gamma(2m+a)^2}, \quad (2.8)$$

where $\mathrm{Re}(c) > 0$, and $\mathrm{Re}(a-c) > -1$. By use of Stirling's asymptotic expression for the Γ-function, and the integral $\int_0^\pi \ln \sin\theta \, d\theta = -\pi \ln 2$, one can show that the exact results (2.7-8) are in agreement with the asymptotic result (2.5).

On the interval 0 to $+\infty$ we have the (monic) Laguerre polynomials. Here the measure is $x^a e^{-x}$, and the normalization constants are,

$$N(m) = \Gamma(m+1)\Gamma(a+m+1). \tag{2.9}$$

The last set of classical orthogonal polynomials on our short list is the (monic) Hermite polynomials. They are orthogonal over the interval $-\infty$ to $+\infty$ with the measure e^{-x^2}. The normalization result is given by,

$$N(m) = \frac{m!\sqrt{\pi}}{2^m}. \tag{2.10}$$

In the present context, the point of formulae (2.7-10) is that they give the exact result for $D(m)$ as a function of m. From them we can, by technique, deduce the asymptotic results for $D(m)$, including at least the leading order terms in an asymptotic expansion in m^{-1}. For the case of (2.10), we are lead, by the classical results to the starting point of problem (1.1-2) where all the $g_p = 0$. Also see (2.12) below for an alternate derivation.

Next we recall the results of Selberg's integral. References here are Habsieger's thesis [17] and Mehta [10]. Specifically, Selberg's [18] integral is,

$$\int_0^1 \cdots \int \prod_{i=1}^{n} t_i^{x-1}(1-t_i)^{y-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2z} \, dt_1 \ldots dt_n$$
$$= \prod_{j=0}^{n-1} \frac{\Gamma(x+jz)\Gamma(y+jz)\Gamma((j+1)z+1)}{\Gamma(x+y+(n+j-1)z)\Gamma(z+1)}. \tag{2.11}$$

Note is taken that this integral corresponds to the Jacobi polynomials with $c = x$ and $a = x+y-1$, and it is also possible to derive (2.11) for $z = 1$ from (2.8). Bombieri and Selberg [17], make the substitutions in (2.11) $x = y$ and $2t_i = 1 + \frac{s_i}{\sqrt{2(x-1)}}$ and let $x \to \infty$ which yields the result,

$$\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{+\infty} \cdots \int \exp\left(-\frac{1}{2}\sum_{i=1}^{n} s_i^2\right) \prod_{1\le i<j\le n} |s_i - s_j|^{2z} \, ds_1 \ldots ds_n$$
$$= \prod_{j=1}^{n} \frac{\Gamma(jz+1)}{\Gamma(z+1)}. \tag{2.12}$$

This result could also have been derived for z=1 from (2.10) as noted above. Again we get an explicit formula for $D(n)$ which lead directly to its asymptotic behavior.

It is of interest from the point of view of the problem (1.1-2) to write out the intermediate step between (2.11) and (2.12). It is,

$$\frac{1}{4^{n(x-1)}}\int_{-\sqrt{2(x-1)}}^{\sqrt{2(x-1)}}\cdots\int\left[\frac{1}{2\sqrt{2(x-1)}}\right]^{zn(n-1)+n}$$
$$\times\exp\left[(x-1)\sum_{i=1}^{n}\ln\left(1-\frac{s_i^2}{2(x-1)}\right)\right]\prod_{1\le i<j\le n}|s_i-s_j|^{2z}\prod_{i=1}^{n}ds_i$$
$$=\frac{1}{4^{nN/8\Lambda}}\left(\frac{\Lambda}{N}\right)^{\frac{1}{2}zn(n-1)+\frac{1}{2}n}\int_{-\frac{1}{2}\sqrt{\frac{N}{\lambda}}}^{\frac{1}{2}\sqrt{\frac{N}{\lambda}}}\cdots\int$$
$$\times\exp\left\{\sum_{i=1}^{n}\left[-\frac{s_i^2}{2}-\sum_{j=2}^{\infty}\frac{(4s_i^2\Lambda)^j}{8\Lambda jN^{j-1}}\right]\right\}\prod_{1\le i<j\le n}|s_i-s_j|^{2z}\prod_{i=1}^{n}ds_i$$
$$=\prod_{j=0}^{n-1}\frac{[\Gamma(\frac{N}{8\Lambda}+1+jz)]^2\Gamma((j+1)z+1)}{\Gamma(2+\frac{N}{4\Lambda}+(n+j-1)z)\Gamma(z+1)},\tag{2.13}$$

where the substitution $x=1+[N/(8\Lambda)]$ was made. Now (2.12) results from (2.13) when the limit $N/\Lambda\to\infty$ is taken. If instead in this form, we identify N with n and Λ with the λ of (1.6), the results of (2.13) mimic the form (1.2) with the

$$g_p=-2^{2p-3}(-\lambda)^{p-1}/p,\tag{2.14}$$

with the interval of integration cut off at $\pm\frac{1}{2}\sqrt{\frac{N}{\Lambda}}$ instead of going to infinity.

III. Continued Fraction of Gauss

One of the most productive methods of generating exact formulae for $D(m)$ in common cases is based on the continued fraction of Gauss. Its derivation is based on the contiguous relations between the hypergeometric functions. Specifically [15], the identities,

$$\frac{{}_2F_1(\alpha,\beta+1;\gamma+1;x)}{{}_2F_1(\alpha,\beta;\gamma;x)}$$
$$=\frac{1}{1-\left(\frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}x\right)\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+2;x)}{{}_2F_1(\alpha,\beta+1;\gamma+1;x)}},\tag{3.1}$$

and, which is really the same as (3.1) when the symmetry between α and β is noted,

$$\begin{aligned}&\frac{{}_2F_1(\alpha+1,\beta+1;\gamma+2;x)}{{}_2F_1(\alpha,\beta+1;\gamma+1;x)}\\&\quad=\frac{1}{1-\left(\dfrac{(\beta+1)(\gamma-\alpha+1)}{(\gamma+1)(\gamma+2)}x\right)\dfrac{{}_2F_1(\alpha+1,\beta+2;\gamma+3;x)}{{}_2F_1(\alpha+1,\beta+1;\gamma+2;x)}},\end{aligned}\tag{3.2}$$

can be substituted alternately to obtain Gauss's continued fraction,

$$\frac{{}_2F_1(\alpha,\beta+1;\gamma+1;x)}{{}_2F_1(\alpha,\beta;\gamma;x)}\equiv G(x)=\cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2}{1+\cfrac{a_3x}{1+\ddots}}}},\tag{3.3}$$

where we have,

$$a_{2n+1}=-\frac{(\alpha+n)(\gamma-\beta+n)}{(\gamma+2n)(\gamma+2n+1)},\quad a_{2n}=-\frac{(\beta+n)(\gamma-\alpha+n)}{(\gamma+2n-1)(\gamma+2n)}.\tag{3.4}$$

If we hypothesize that $G(x)$ of (3.3) is of the form,

$$G(x)=\int_0^1\frac{d\phi(u)}{1-ux}=\sum_{j=0}^{\infty}g_jx^j,\tag{3.5}$$

then the Taylor's series coefficients are of the form

$$g_j=\int_0^1 u^j\,d\phi(u),\tag{3.6}$$

the moments of a distribution. We will show presently that this hypothesis is correct. In the theory of continued fractions [19], it is known that,

$$a_{2m-1}=-\frac{D(m-1)E(m)}{D(m)E(m-1)},\quad a_{2m}=-\frac{D(m+1)E(m-1)}{D(m)E(m)},\tag{3.7}$$

where $D(m)$ is as defined by (2.1), and

$$E(m)=\det\parallel g_{i+j-1}\parallel_{1\le i,j\le m}\,.\tag{3.8}$$

The combinations

$$r_m \equiv a_{2m-1}a_{2m} = \frac{D(m-1)D(m+1)}{[D(m)]^2}, \tag{3.9}$$
$$s_m \equiv a_{2m}a_{2m+1} = \frac{E(m-1)E(m+1)}{[E(m)]^2},$$

allow the separation of the D and E determinants to be made directly from the continued fraction coefficients a_m. If we use the easily computed result, $a_0 = g_0$, then we get simply from (3.9),

$$\frac{D(m+1)}{D(m)} = g_0 \prod_{j=1}^{m} r_j. \tag{3.10}$$

and hence,

$$D(m+1) = g_0^{m+1} \prod_{j=1}^{m} r_j^{m-j+1}. \tag{3.11}$$

Now from (3.4) and (3.9) we can compute that,

$$r_m = \frac{(\beta+m)(\gamma-\alpha+m)(\alpha+m-1)(\gamma-\beta+m-1)}{(\gamma+2m)(\gamma+2m-1)^2(\gamma+2m-2)}, \tag{3.12}$$

and that $g_0 = 1$, so that we get from (3.10),

$$\frac{D(m+1)}{D(m)} = \tag{3.13}$$
$$\frac{\Gamma(\beta+m+1)\Gamma(\gamma-\alpha+m+1)\Gamma(\alpha+m)\Gamma(\gamma-\beta+m)\Gamma(\gamma+1)\Gamma(\gamma)}{\Gamma(\beta+1)\Gamma(\gamma-\alpha+1)\Gamma(\alpha)\Gamma(\gamma-\beta)\Gamma(\gamma+2m+1)\Gamma(\gamma+2m)},$$

and from (3.11)

$$D(m+1) = \tag{3.14}$$
$$\prod_{k=1}^{m} \frac{\Gamma(\beta+k+1)\Gamma(\gamma-\alpha+k+1)\Gamma(\alpha+k)\Gamma(\gamma-\beta+k)\Gamma(\gamma+1)\Gamma(\gamma)}{\Gamma(\beta+1)\Gamma(\gamma-\alpha+1)\Gamma(\alpha)\Gamma(\gamma-\beta)\Gamma(\gamma+2k+1)\Gamma(\gamma+2k)},$$

As long as

$$\alpha > 0, \quad \beta > -1, \quad \gamma > 0, \quad \gamma-\alpha > -1, \quad \gamma-\beta > 0, \tag{3.15}$$

all the determinants $D(m), E(m) \geq 0$, which is a necessary and sufficient condition that form (3.6) be valid with $d\phi(u) \geq 0$, but on an infinite range. Since the hypergeometric functions are known [20] to be analytic in the cut plane $1 \leq x \leq \infty$ and positive for $0 \leq x \leq 1$ if (3.15) holds, the range of the integral in (3.6) is thereby restricted to that given, and that form is valid, provided (3.15) holds.

Of special interest for applications to quantum gravity theory are the cases where the measure is over an infinite range. To construct these cases, let us use the confluence,

$$ {}_2F_0(\alpha,\beta;x) = \lim_{\gamma\to\infty} {}_2F_1(\alpha,\beta;\gamma;\gamma x). \tag{3.16} $$

Now the cut in the resultant function will be $0 \leq x \leq \infty$. These functions have the integral representations,

$$ {}_2F_0(\alpha,\beta;-x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}e^{-t}\,dt}{(1+tx)^\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty \frac{t^{\beta-1}e^{-t}\,dt}{(1+tx)^\alpha}. \tag{3.17} $$

The continued fraction we obtain in this case is,

$$ \frac{{}_2F_0(\alpha,\beta+1;-x)}{{}_2F_0(\alpha,\beta;-x)} \equiv \hat{G}(x) = \cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2}{1+\cfrac{a_3x}{1+\ddots}}}}, \tag{3.18} $$

where

$$ a_{2n+1} = \alpha+n, \quad a_{2n} = \beta+n. \tag{3.19} $$

The normalization factor (3.13) here reduces to,

$$ \frac{D(m+1)}{D(m)} = \frac{\Gamma(\beta+m+1)\Gamma(\alpha+m)}{\Gamma(\beta+1)\Gamma(\alpha)}, \tag{3.20} $$

and the result for the determinants is,

$$ D(m+1) = \prod_{j=1}^{m}\left\{\frac{\Gamma(\beta+1+j)\Gamma(\alpha+j)}{\Gamma(\beta+1)\Gamma(\alpha)}\right\}. \tag{3.21} $$

The special case where $\beta = 0$ in (3.18) gives us directly the problem for the Laguerre polynomials (2.9) with $a = \alpha - 1$, when we remember that the normalization of the measure is changed by a factor of $1/\Gamma(\alpha)$ here from that in section 2. These determinants (and the corresponding $E(m)$) are all positive so long as

$$ \alpha > 0, \quad \beta > -1, \tag{3.22} $$

so this ratio of confluent hypergeometric functions is of the form (3.5) with the range of integration 0 to ∞, provide that (3.22) holds.

According to Stieltjes inversion formula [15] the integrated measure $\phi(u)$ is given by,

$$\phi(u) - \phi(v) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_v^u \mathrm{Im} \left\{ \hat{G}\left[-(w - i\epsilon)^{-1}\right] \right\} \frac{dw}{w}, \tag{3.23}$$

where $\phi(\infty) - \phi(0) = g_0$. The integral representations (3.17) can be used to deduce the function ϕ for this case, which will be non-negative definite, again provided (3.22) holds.

IV. The q-Extension of the Continued Fraction of Gauss

The necessary extension for this section is that of the hypergeometric functions to the so-called basic hypergeometric functions [21, 22], or the q-extension. The series expansion for the hypergeometric functions is,

$$_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1!\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \cdots + \frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n}x^n + \cdots, \tag{4.1}$$

where

$$\begin{aligned} &(a)_n \equiv a(a+1)\cdots(a+n-1), \quad n \geq 1 \\ &(a)_0 \equiv 1. \end{aligned} \tag{4.2}$$

The basis of the identities in the (3.1-2) on which the results of the previous section were based are the contiguous relations, *e.g.*,

$$\begin{aligned} &_2F_1(\alpha, \beta; \gamma; x) -{}_2F_1(\alpha, \beta+1; \gamma+1; x) \\ &\qquad = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)} x\, {}_2F_1(\alpha+1, \beta+1; \gamma+2; x), \end{aligned} \tag{4.3}$$

as can be seen by use of the series expansion (4.1) and term by term manipulation in the region of convergence ($|x| < 1$). In order to make the q-extension of this identity, we introduce the q-version of numbers,

$$[a]_q = \frac{1-q^a}{1-q}, \quad \lim_{q\to 1}[a]_q = a. \tag{4.4}$$

In terms of this notation, the definition of the q-extension of the hypergeometric functions, which goes back at least to Heine [23], is,

$$_2F_1^{(q)}(\alpha, \beta; \gamma; x) = 1 + \sum_{n=1}^{\infty} \frac{([\alpha]_q)_n([\beta]_q)_n}{([1]_q)_n([\gamma]_q)_n} x^n, \tag{4.5}$$

where now,

$$
\begin{aligned}
&([a]_q)_n \equiv [a]_q[a+1]_q \cdots [a+n-1]_q, \quad n \geq 1 \\
&([a]_q)_0 \equiv 1.
\end{aligned}
\tag{4.6}
$$

The contiguous relation (4.3) extends to

$$
\begin{aligned}
{}_2F_1^{(q)}(\alpha,\beta;\gamma;x) - {}_2F_1^{(q)}(\alpha,\beta+1;\gamma+1;x) \\
= \frac{[\alpha]_q([\gamma]_q - [\beta]_q)}{[\gamma]_q[\gamma+1]_q} x\, {}_2F_1^{(q)}(\alpha+1,\beta+1;\gamma+2;x),
\end{aligned}
\tag{4.7}
$$

which gives the required identity, as can again be seen by use of the series expansion (4.5) and term by term manipulation in the region of convergence. As one follows through the algebra involved in these manipulations, one finds that the property,

$$
[x]_q[y+z]_q - [x+z]_q[y]_q \equiv q^y[z]_q[x-y]_q, \tag{4.8}
$$

is, by itself, sufficient to obtain the contiguous relation (4.7). A through investigation of necessary and sufficient conditions for the extension of hypergeometric functions which retain the contiguous relations will be given in the next section.

If we now follow the analysis of the previous section with the coefficient in (4.7) replacing appropriately that in (3.1), we may derive the continued fraction,

$$
\frac{{}_2F_1^{(q)}(\alpha,\beta+1;\gamma+1;x)}{{}_2F_1^{(q)}(\alpha,\beta;\gamma;x)} \equiv G^{(q)}(x) = \cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2}{1+\cfrac{a_3x}{1+\ddots}}}}, \tag{4.9}
$$

where we have,

$$
\begin{aligned}
a_{2n+1} &= -\frac{[\alpha+n]_q([\gamma+2n]_q - [\beta+n]_q)}{[\gamma+2n]_q[\gamma+2n+1]_q}, \\
a_{2n} &= -\frac{[\beta+n]_q([\gamma+2n-1]_q - [\alpha+n-1]_q)}{[\gamma+2n-1]_q[\gamma+2n]_q}.
\end{aligned}
\tag{4.10}
$$

At this point it is of interest to take note of the analytic nature of the q-extension of the hypergeometric functions. In the limit of large n, we deduce from (4.10) that

$$
\begin{aligned}
a_{2n+1} \asymp -q^{\alpha-\gamma-1-n} \underset{n\to\infty}{\to} 0, \quad a_{2n} \asymp -q^{\beta-\gamma-n} \underset{n\to\infty}{\to} 0, \quad |q| > 1, \\
a_{2n+1} \asymp -q^{\beta+n} \underset{n\to\infty}{\to} 0, \quad a_{2n} \asymp -q^{\alpha+n-1} \underset{n\to\infty}{\to} 0, \quad |q| < 1,
\end{aligned}
\tag{4.11}
$$

By a theorem of Jones and Thron [24] these continued fractions ($|q| \neq 1$) converge to meromorphic functions of x and are regular at $x = 0$. Nevertheless we know that when $q \to 1$ we get a function with a branch cut, $1 \leq x \leq \infty$. If q is a root of unity other than $q = 1$, a periodic continued fraction results and the exact solution can be found for $G^{(q)}(x)$.

In order to deduce the behavior of the Toeplitz determinants, as at (3.9 and 12) we compute,

$$r_m = \frac{[\beta+m]_q([\gamma+2m-1]_q - [\alpha+m-1]_q)[\alpha+m-1]_q}{[\gamma+2m-2]_q[\gamma+2m-1]_q^2[\gamma+2m]_q} \times ([\gamma+2m-2]_q - [\beta+m-1]_q). \tag{4.12}$$

In order to conveniently express the result for the determinants, if is useful next to introduce the q-extension of the Γ-function. This extension is customarily given in terms of infinite products. Let us define,

$$(z)_\infty \equiv \prod_{j=0}^{\infty}(1 - zq^j), \quad |z| < 1, \quad |q| < 1, \tag{4.13}$$

where it is easy to show that for the given range of variable that the infinite product converges. We may now define,

$$\Gamma^{(q)}(x) \equiv \frac{(q)_\infty (1-q)^{x-1}}{(q^x)_\infty}. \tag{4.14}$$

Some properties of this definition are,

$$\Gamma^{(q)}(y+n) = \left[\prod_{j=0}^{n-1}[y+j]_q\right]\Gamma^{(q)}(y), \quad \Gamma^{(q)}(2) = 1,$$
$$\Gamma^{(q)}(n+1) = \prod_{j=1}^{n}[j]_q \underset{q\to 1^-}{\to} n!. \tag{4.15}$$

Thus we may write

$$\frac{D(m+1)}{D(m)} = q^{(\alpha+\beta-1)m+m(m+1)} \times \frac{\Gamma^{(q)}(\beta+m+1)\Gamma^{(q)}(\gamma-\alpha+m+1)\Gamma^{(q)}(\alpha+m)}{\Gamma^{(q)}(\beta+1)\Gamma^{(q)}(\gamma-\alpha+1)\Gamma^{(q)}(\alpha)} \times \frac{\Gamma^{(q)}(\gamma-\beta+m)\Gamma^{(q)}(\gamma+1)\Gamma^{(q)}(\gamma)}{\Gamma^{(q)}(\gamma-\beta)\Gamma^{(q)}(\gamma+2m+1)\Gamma^{(q)}(\gamma+2m)}, \tag{4.16}$$

where use was made of the identity,

$$[x]_q - [y]_q = q^y[x-y]_q, \tag{4.17}$$

to write the result in this factored form. The value for the Toeplitz determinant is

$$\begin{aligned} D(m+1) = {} & q^{\frac{1}{2}(\alpha+\beta-1)m(m+1)+\frac{1}{3}m(m+1)(m+2)} \\ & \times \prod_{k=1}^{m}\left[\frac{\Gamma^{(q)}(\beta+k+1)\Gamma^{(q)}(\gamma-\alpha+k+1)\Gamma^{(q)}(\alpha+k)}{\Gamma^{(q)}(\beta+1)\Gamma^{(q)}(\gamma-\alpha+1)\Gamma^{(q)}(\alpha)}\right. \\ & \qquad \left. \times \frac{\Gamma^{(q)}(\gamma-\beta+k)\Gamma^{(q)}(\gamma+1)\Gamma^{(q)}(\gamma)}{\Gamma^{(q)}(\gamma-\beta)\Gamma^{(q)}(\gamma+2k+1)\Gamma^{(q)}(\gamma+2k)}\right], \end{aligned} \tag{4.18}$$

We can also consider the q-extension of the confluent hypergeometric function. Here we have, for

$${}_2F_0^{(q)}(\alpha,\beta;x) = 1 + \sum_{n=1}^{\infty}\frac{([\alpha]_q)_n([\beta]_q)_n}{([1]_q)_n}x^n, \tag{4.19}$$

the corresponding continued fraction,

$$\frac{{}_2F_0^{(q)}(\alpha,\beta+1;x)}{{}_2F_1^{(q)}(\alpha,\beta;x)} \equiv \hat{G}^{(q)}(x) = \cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2}{1+\cfrac{a_3x}{1+\ddots}}}}, \tag{4.20}$$

with

$$a_{2n+1} = -q^{\beta+n}[\alpha+n]_q, \quad a_{2n} = -q^{\alpha+n-1}[\beta+n]_q. \tag{4.21}$$

In this case, the asymptotic properties of the coefficients are

$$\begin{gathered} a_{2n+1} \asymp -\frac{q^{\alpha+\beta+2n}}{q-1} \underset{n\to\infty}{\to} \infty, \quad a_{2n} \asymp -\frac{q^{\beta+\alpha+2n-1}}{q-1} \underset{n\to\infty}{\to} \infty, \quad |q|>1, \\ a_{2n+1} \asymp -\frac{q^{\beta+n}}{1-q} \underset{n\to\infty}{\to} 0, \quad a_{2n} \asymp -\frac{q^{\alpha+n-1}}{1-q} \underset{n\to\infty}{\to} 0, \quad |q|<1, \end{gathered} \tag{4.22}$$

In the case $|q|<1$ we have as before the conclusion that $\hat{G}^{(q)}(x)$ is a meromorphic function which is regular at the origin. In the case $|q|>1$, however we have divergent coefficients in the continued fraction reflecting the divergent Taylor series.

V. Generalized Hypergeometric Functions or the Uniqueness of the q-extension

In this section we consider in detail the question of whether there are further generalizations of the hypergeometric function, besides the q-extension, which preserve the existence of contiguous relations. We will find that such is not the case. It is known [25] that the q-extension of the hypergeometric function has a deep and intimate connection with the theory of quantum groups, and as such our results have, we feel, important implications there as well but we will not explore them in this paper. The theory of quantum groups in turn arose from the study of solvable models in statistical mechanics, and has now been shown to have applications in a number of other areas.

We, in the spirit of the q-extension replace each number x by a mapping of the complex plane, $T(x)$, so that the generalization we consider becomes,

$$
{}_2F_1^T(\alpha,\beta;\gamma;x) = 1 + \frac{T(\alpha)T(\beta)}{T(\gamma)T(1)}x + \frac{T(\alpha)T(\alpha+1)T(\beta)T(\beta+1)}{T(\gamma)T(\gamma+1)T(1)T(2)}x^2 + \cdots + \prod_{j=0}^{n-1}\frac{T(\alpha+j)T(\beta+j)}{T(\gamma+j)T(1+j)}x^n + \cdots \tag{5.1}
$$

If we compute then the proposed contiguous relation,

$$
{}_2F_1^T(\alpha,\beta+1;\gamma+1;x) - {}_2F_1^T(\alpha,\beta;\gamma;x) = x\frac{T(\alpha)}{T(1)}\left[\frac{T(\beta+1)}{T(\gamma+1)} - \frac{T(\beta)}{T(\gamma)}\right] + \cdots + x^n\prod_{j=0}^{n-1}\frac{T(\alpha+j)T(\beta+j)}{T(\gamma+j)T(1+j)}\left[\frac{T(\beta+n)}{T(\gamma+n)} - \frac{T(\beta)}{T(\gamma)}\right] + \cdots . \tag{5.2}
$$

We suppose that

$$
\frac{1}{T(n)}\left[\frac{T(\beta+n)}{T(\gamma+n)} - \frac{T(\beta)}{T(\gamma)}\right] = \frac{C(\beta,\gamma)}{T(\gamma+n)T(\gamma)}, \tag{5.3}
$$

where $C(\beta,\gamma)$ is independent of n so that we may write,

$$
{}_2F_1^T(\alpha,\beta+1;\gamma+1;x) - {}_2F_1^T(\alpha,\beta;\gamma;x) = x\frac{T(\alpha)C(\beta,\gamma)}{T(\gamma)T(\gamma+1)}\,{}_2F_1^T(\alpha+1,\beta+1;\gamma+2;x). \tag{5.4}
$$

As in the preceding sections we would then get,

$$\frac{{}_2F_1^T(\alpha,\beta+1;\gamma+1;x)}{{}_2F_1^T(\alpha,\beta;\gamma;x)} \equiv R(\alpha,\beta;\gamma;x) = \cfrac{1}{1+\cfrac{a_1x}{1+\cfrac{a_2}{1+\cfrac{a_3x}{1+\ddots}}}}, \quad (5.5)$$

where

$$\begin{aligned} a_{2n+1} &= -\frac{T(\alpha+n)C(\beta+n,\gamma+2n)}{T(\gamma+2n)T(\gamma+2n+1)}, \\ a_{2n+2} &= -\frac{T(\beta+n+1)C(\alpha+n,\gamma+2n+1)}{T(\gamma+2n+1)T(\gamma+2n+2)}, \quad n\geq 0. \end{aligned} \quad (5.6)$$

If we use the case $T(x)=x$, we obtain, $C(\beta,\gamma)=\gamma-\beta$ which leads to (3.4) directly. As a second example, if we select $T(x)=[x]_q$ as defined at (4.4) then we get $C(\beta,\gamma)=[\gamma]_q-[\beta]_q$ and (4.10) follows directly from (5.6).

Our condition (5.3) can be rewritten as,

$$T(x)T(y+z)-T(y)T(x+z)=C(y,x)T(z), \quad (5.7)$$

which is to be compared with (4.8) for the q-extension case. We will suppose in what follows that T does not vanish identically. If we set $z=0$ in (5.7), then we get that $C(y,x)T(0)=0$. If C does not vanish identically, then we conclude that,

$$T(0)=0. \quad (5.8)$$

On the other hand if $C\equiv 0$, then (5.7) would imply that

$$T(x)T(y+z)=T(y)T(x+z), \text{ or } \frac{T(x+z)}{T(x)}=\frac{T(y+z)}{T(y)}=Q(z), \quad (5.9)$$

where Q is independent of x and y. One solution in this case is $T(x)=AB^x$. The case $C\equiv 0$ means by (5.6) that the continued fraction R of (5.5) is just $R=1$, the trivial non-interesting case. Therefore we will hence forth treat just the case $C\not\equiv 0$.

It is easy to see that $C(x,y)=-C(y,x)$ from (5.7). If we set $x=0$ in (1), then we conclude that,

$$-T(y)T(z)=C(y,0)T(z), \quad \Rightarrow \quad C(y,0)=-T(y), \quad (5.10)$$

as we have assumed that $T(z) \not\equiv 0$. Next set $z=-y$ in (5.7). We get,

$$C(y,x)T(-y) = T(x)T(0) - T(y)T(x-y) = -T(y)T(x-y), \quad (5.11)$$

by (5.8). If we multiply (5.7) by $T(-y)$ and substitute in (5.11) we have,

$$T(x)T(y+z)T(-y) - T(x+z)T(y)T(-y) + T(y)T(x-y)T(z) = 0, \quad (5.12)$$

which involves only the function T and not the auxiliary function C, which is given in terms of T (provided $T(-y) \neq 0$), by

$$C(y,x) = -\frac{T(y)}{T(-y)}T(x-y). \quad (5.13)$$

Thus (5.7) is equivalent to two cases (i) (5.9) which is of no interest, and (5.12), with the auxiliary function normally given by (5.13). It is convenient to re-write by a simple linear change of variables (5.12) in the more symmetrical, equivalent form,

$$T(x-z)T(y)T(z-y) - T(x)T(y-z)T(z-y) + T(y-z)T(x-y)T(z) = 0. \quad (5.14)$$

The next step is the reduction of (5.14). Note that the substitution $z=0$ once again yields (5.8). To accomplish the reduction, we first set $x=0$, which yields,

$$\left(-\frac{T(y)}{T(-y)}\right)\left(-\frac{T(z-y)}{T(y-z)}\right) = -\frac{T(z)}{T(-z)}, \quad (5.15)$$

provided that none of the denominators vanish. In turn we may write (5.15) as

$$g(x+y) = g(x)g(y), \quad (5.16)$$

where we define,

$$g(x) = -\frac{T(x)}{T(-x)}. \quad (5.17)$$

We recognize (5.16) as the Cauchy multiplicative equation. The only locally bounded solutionns of this equation are,

$$g(x) = q^x, \quad (5.18)$$

for some complex number q.

More generally from here on we will assume that $g(x)$ satisfies (5.16) but is not necessarily bounded. $g(x_0)$ can never vanish because we would then have $g(x_0+y)=0\,\forall y$ and hence $g(x)\equiv 0$ by (5.16). Thus as $g(0)=g(0)^2$, it must be that $g(0)=1$. Let $\gamma(x)$ be an arbitrary solution of (5.16). Then, if $T(x)$ satisfies (5.14) with $g(x)$ given by (5.17), it follows that $\tilde{T}(x)=T(x)\gamma(x)$ also satisfies (5.14), and

$$\tilde{g}(x)=-\frac{T(x)\gamma(x)}{T(-x)\gamma(-x)}=g(x)\gamma(2x), \tag{5.19}$$

since $\gamma(2x)\gamma(-x)=\gamma(x)$. If we select,

$$\gamma(2x)=g(-x)=\frac{1}{g(x)}, \tag{5.20}$$

then we see that $\tilde{T}(x)$ satisfies (5.14) with $\tilde{g}(x)=1$. From this result we conclude by division of (5.14) by $T(z-y)$, that the general solution of (5.14) is of the form,

$$T(x)=g(x)t(x), \tag{5.21}$$

where $g(x)$ satisfies (5.16) and $t(x)$ satisfies the equation,

$$t(x-z)t(y)-t(x)t(y-z)=t(z)t(x-y), \tag{5.22}$$

for the particular case $g=1$, *i.e.*, $t(y)=-t(-y)$.

Now it remains to analyze the solutions of (5.22). In particular (5.22) again implies from the $z=0$ case that $t(0)=0$, and the g restriction that $t(x)$ is odd follows by an interchange of x and y. Using this anti-symmetry, we can rewrite (5.22) in the form,

$$t(x)t(y-z)+t(y)t(z-x)+t(z)t(x-y)=0, \tag{5.23}$$

which is invariant under the circular permutation $(x,y,z)\mapsto(z,x,y)$. With the condition (5.8), then if we put $x=y$ in (5.23), we can conclude directly from (5.23) that $t(x)$ is odd so we can go back from (5.23) to (5.22), so the two equations are equivalent. In fact, $t(0)\neq 0$ is inconsistent with (5.23). This result can be seen by setting $s=t(0)$, then (5.23) for the case $x=y$ becomes,

$$-st(z)=t(x)\{t(x-z)+t(z-x)\},\quad -st(x)=t(z)\{t(x-z)+t(z-x)\}, \tag{5.24}$$

where the second equation is the x, z interchange of the first. If we multiply the equations by $t(z)$ and $t(x)$ respectively, subtract and divide by s, assumed non-zero, we get,

$$[t(z)]^2=[t(x)]^2. \tag{5.25}$$

This result implies that $t(z)$ is a constant, up to a sign, and thus is of the form $t(x) = C\epsilon(x)$, where $\epsilon(x) = \pm 1$. Thus, we would be able to rewrite (5.23) as,

$$\epsilon(x)\epsilon(y-z) + \epsilon(y)\epsilon(z-x) + \epsilon(z)\epsilon(x-y) = 0, \tag{5.26}$$

which is obviously impossible. Thus we have a proof by contradiction that $t(0) = 0$ and can conclude that (5.22) and (5.23) are equivalent.

The next step is to analyze the solutions of (5.23). It is worthwhile to notice, and it will serve as a guide to our analysis, that $t(x) = A\sinh(Bx)$ satisfies (5.23), as can be seen through the use of standard hyperbolic function identities! In general we can deduce from (5.23), by letting $y = -x$ and using the anti-symmetry property that

$$t(2x)t(z) = t(x)\{t(x+z) + t(z-x)\}, \tag{5.27}$$

If we now define,

$$U(x) \equiv \frac{t(2x)}{2t(x)}, \tag{5.28}$$

which would be the normalized cosh if $t(x)$ is the unnormalized sinh, then we get the result from (5.27),

$$U(x) = \frac{t(z+x) + t(z-x)}{2t(z)}. \tag{5.29}$$

Note is made that as t is odd, by (5.28) U is even. We can rewrite (5.29) in two ways related by an x-z interchange as,

$$t(z+x) = 2t(z)U(x) - t(z-x), \quad t(z+x) = 2t(x)U(z) - t(x-z). \tag{5.30}$$

If we add these two equations and use the anti-symmetry property, then we get,

$$t(z+x) = t(z)U(x) + t(x)U(x), \tag{5.31}$$

which is the direct analogue of the addition formula for the sinh.

Now let us consider the quantity,

$$D = \frac{U(x+y) - U(x)U(y)}{t(x)} - \frac{U(x'+y) - U(x')U(y)}{t(x')}. \tag{5.32}$$

If we use (5.29) then we can write,

$$\begin{aligned} U(x+y) &= \frac{t(x+x'+y) + t(x'-x-y)}{2t(x')} \\ U(x'+y) &= \frac{t(x+x'+y) + t(x-x'-y)}{2t(x)}, \end{aligned} \tag{5.33}$$

which when substituted into (5.32) gives,

$$D = \frac{t(x'-x-y) - t(x-x'-y) - 2U(y)[U(x)t(x') - U(x')t(x)]}{2t(x')t(x)}. \tag{5.34}$$

By the addition formula (5.31) the square bracket term reduces to $t(x - x')$, so that we get,

$$D = \frac{t(x'-x-y) - t(x-x'-y) - 2U(y)t(x'-x)}{2t(x')t(x)}. \tag{5.35}$$

Now if we use (5.29) to express

$$U(y) = \frac{t(x'-x+y) + t(x'-x-y)}{2t(x'-x)}, \tag{5.36}$$

and the anti-symmetry property of t, we find that $D = 0$. From this we conclude that

$$\frac{U(x+y) - U(x)U(y)}{t(x)t(y)} \tag{5.37}$$

is independent of x, and hence by symmetry, independent of y as well. Therefore there exists a constant K such that

$$U(x+y) = U(x)U(y) + Kt(x)t(y), \tag{5.38}$$

which is the analogue of the cosh addition formula!

We are now ready to deduce the general solution of the reduced equation (5.23). First suppose that t is odd and fulfills (5.23) and that U is given from it by (5.28) so that (5.31) and (5.38) hold. The first case is $K = 0$. Then $U(x+y) = U(x)U(y) = U(x-y)$, so that U is a constant. Thus

$$t(x+y) = U[t(x) + t(y)], \tag{5.39}$$

which implies that $U = 1$ from the case $y = 0$, and so for this case,

$$t(x+y) = t(x) + t(y), \tag{5.40}$$

which is the additive Cauchy equation. Therefore any solution of the additive Cauchy equation satisfies (5.23) as can be verified by direct substitution of (5.40) into (5.23).

In the second case, K is real and positive. If we scale $\mathcal{T}(x) = \sqrt{K}t(x)$, then (5.31) and (5.38) become,

$$\begin{aligned} \mathcal{T}(x+y) &= U(x)\mathcal{T}(y) + U(y)\mathcal{T}(x), \\ U(x+y) &= U(x)U(y) + \mathcal{T}(x)\mathcal{T}(y), \end{aligned} \tag{5.41}$$

and if we take,

$$f(x) = \mathcal{T}(x) + U(x), \tag{5.42}$$

we verify from (5.41) that

$$f(x+y) = f(x)f(y), \tag{5.43}$$

which is again the Cauchy multiplicative equation. Since $\mathcal{T}$ is odd and U is even, we have

$$\begin{aligned} 2\mathcal{T}(x) &= f(x) - f(-x), \\ 2U(x) &= f(x) + f(-x). \end{aligned} \tag{5.44}$$

So the solution of (5.23) in this case is

$$t(x) = C(f(x) - f(-x)), \tag{5.45}$$

where f is an arbitrary solution of the Cauchy multiplicative equation and C is an arbitrary constant.

The third case is when K is real and negative, and we just repeat the analysis of the second case, but in a manner analogous to the case of circular functions, instead of hyperbolic functions. The only distinction between these two cases occurs if there are reality conditions on t. Finally the fourth case is when K is complex, and a complex square root results in the definition of $\mathcal{T}$.

Summarizing, then the solution of the reduced equation (5.23) is

$$\begin{aligned} &t(x) = C\tau(x) \\ &\text{or} \\ &t(x) = C[f(x) - f(-x)] \end{aligned} \tag{5.46}$$

where τ is a solution of the Cauchy additive equation (5.40), and f is a solution of the Cauchy multiplicative equation, (5.43). The general solution of the original equation (5.12) is

$$T(x) = Ch(x)\tau(x), \tag{5.47}$$

or

$$T(x) = Ch(x)[f(x) - f(-x)], \tag{5.48}$$

where τ is an arbitrary solution of (5.40) and h and f are arbitrary solutions of (5.43). The auxiliary function $C(x, y)$ can be computed from the solution T of (5.47-8) by means of (5.13) as remarked above.

It is a classical result by Cauchy [26, 27] that the only locally bounded solutions of (5.40) and (5.43) are Cx and q^x respectively. (Boundedness on any arbitrarily small open set is sufficient.) In this case the only generalization of the hypergeometric functions that occur are just argument-scaled versions of q-extension discussed in section 4. That this is so can be seen by noticing that the case (5.47) implies

$$ {}_2F_1^T(\alpha,\beta;\gamma;x) = {}_2F_1(\alpha,\beta;\gamma;q^{\alpha+\beta-\gamma-1}x), \tag{5.49} $$

and the case (5.48) implies,

$$ {}_2F_1^T(\alpha,\beta;\gamma;x) = {}_2F_1^{(q^{-2})}(\alpha,\beta;\gamma;(\hat{q}q)^{\alpha+\beta-\gamma-1}x). \tag{5.50} $$

Let us now investigate the consequences of choosing the irregular solutions of the Cauchy equations. For (5.47) we have as the required quantities,

$$ \begin{array}{cc} \tau(n) = n\tau(1) = n\tau, & h(n) = [h(1)]^n = h^n, \\ \tau(a+n) = \tau(a) + n\tau(1) = \alpha + n\tau, & \tau(b+n) = \beta + n\tau, \\ h(a+n) = h(a)[h(1)]^n = Ah^n, & \tau(c+n) = \gamma + n\tau, \\ h(b+n) = Bh^n, & h(c+n) = Ch^n. \end{array} \tag{5.51} $$

The general term in the series (5.1) becomes,

$$ \left(\frac{AB}{C}\right)^n \frac{\frac{\alpha}{\tau}\left(\frac{\alpha}{\tau}+1\right)\cdots\left(\frac{\alpha}{\tau}+n-1\right)\frac{\beta}{\tau}\left(\frac{\beta}{\tau}+1\right)\cdots\left(\frac{\beta}{\tau}+n-1\right)}{\frac{\gamma}{\tau}\left(\frac{\gamma}{\tau}+1\right)\cdots\left(\frac{\gamma}{\tau}+n-1\right)n!} z^n, \tag{5.52} $$

which we identify as the general term of ${}_2F_1$ as given by (4.1) where,

$$ z \mapsto \frac{AB}{C}z, \quad \alpha \mapsto \frac{\alpha}{\tau}, \quad \beta \mapsto \frac{\beta}{\tau}, \text{ and } \gamma \mapsto \frac{\gamma}{\tau}. \tag{5.53} $$

For the case (5.48), the result of the factor h is just to rescale z as in (5.53), so with out loss of generality, we can treat the case $h = 1$. Now for this case the required quantities are,

$$ \begin{gathered} f(a+n) = \alpha f^n, \quad f(b+n) = \beta f^n, \quad f(c+n) = \gamma f^n, \quad f = f(1), \\ T(a+n) = \alpha f^n - \frac{1}{\alpha f^n} = \frac{1}{\alpha} f^{-n}[\alpha^2 f^{2n} - 1], \quad T(n) = f^{-n}[f^{2n} - 1]. \end{gathered} \tag{5.54} $$

The general term of (5.1) becomes,

$$\left(\frac{\gamma}{\alpha\beta}\right)^n \frac{(\alpha^2-1)(\alpha^2 f^2-1)\cdots(\alpha^2 f^{2(n-1)}-1)}{(\gamma^2-1)(\gamma^2 f^2-1)\cdots(\gamma^2 f^{2(n-1)}-1)} \times \frac{(\beta^2-1)(\beta^2 f^2-1)\cdots(\beta^2 f^{2(n-1)}-1)}{(f^2-1)(f^4-1)\cdots(f^{2n}-1)} z^n. \tag{5.55}$$

If we set $f^2 = q$, and $\alpha^2 = q^A$, $\beta^2 = q^B$, $\gamma^2 = q^C$ then (5.55) becomes,

$$\frac{(1-q^A)(1-q^{A+1})\cdots(1-q^{A+n-1})}{(1-q^C)(1-q^{C+1})\cdots(1-q^{C+n-1})} \times \frac{(1-q^B)(1-q^{B+1})\cdots(1-q^{B+n-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \left(\frac{\gamma z}{\alpha\beta}\right)^n, \tag{5.56}$$

which is the ordinary q-extension hypergeometric function of section 4 with suitably modified parameters. Hence neither regular nor the irregular solutions of Cauchy's equations bring any new functions and we therefore conclude that the q-extension is unique in the sense that the requirement that there continue to be contiguous relations so restricts the mapping $T(x)$ that no further cases are possible!

As a final remark in this section, we note that in (5.7) we only really needed the case z integer and it is unclear to us whether the assumption that (5.7) is satisfied for any z is or is not superfluous.

VI. Leading Order Asymptotic Behavior by the Saddle Point Method

In this section we will compute the asymptotic behavior of the Toeplitz form $D(N)$ of (1.11) in the form $D(N) = \bar{Z}_N(g)/N!$ as given by (1.10). We will basically use the saddle-point method of Brézin *et al.* [4]. We begin with

$$\bar{Z}_N(g) = \int_{-\infty}^{\infty} \cdots \int \Delta^\gamma(\lambda) \prod_{i=1}^{N} d\mu(\lambda_i), \tag{6.1}$$

where we treat now the more general case of Δ^γ instead of Δ^2. We differentiate (6.1) with respect to the λ_i (1.8) and obtain the saddle point equations,

$$\gamma \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{1}{\lambda_i - \lambda_j} - V'(\lambda_i) = 0, \tag{6.2}$$

where note is taken that the first term in this equation is of the order of N. Let us now suppose that asymptotically as $\lambda \to \infty$,

$$V'(\lambda) \asymp A\lambda^{\alpha}. \tag{6.3}$$

If we organize the λ_i such that,

$$\lambda_i \mapsto N^{\beta}\lambda(x), \quad x = \frac{i}{N}, \tag{6.4}$$

then if we select,

$$\beta = \frac{1}{1+\alpha}, \tag{6.5}$$

we may write the saddle point equations (6.2) in the limit as $N \to \infty$ as,

$$A\lambda^{\alpha}(x) = \gamma \fint_0^1 \frac{dy}{\lambda(x) - \lambda(y)}, \tag{6.6}$$

where $\fint$ is the principal value integral. The next step is to introduce the function,

$$\frac{dx}{d\lambda} = u(\lambda) \geq 0, \tag{6.7}$$

by construction. In addition, by construction, we must have the normalization condition,

$$\int_{-2a}^{2a} u(\lambda)\, d\lambda = 1. \tag{6.8}$$

We may rewrite (6.6) as an equation for $u(\lambda)$,

$$A\lambda^{\alpha} = \gamma \fint_{-2a}^{2a} \frac{u(\mu)\, d\mu}{\lambda - \mu}, \text{ for } -2a \leq \lambda \leq 2a, \tag{6.9}$$

where for $\alpha \geq 0$, we must have finite support for $u(\mu)$ or the equation would be inconsistent for large λ.

Let us now introduce, for complex λ, the analytic function,

$$\begin{aligned} F(\lambda) &= \int_{-2a}^{2a} \frac{u(\mu)\, d\mu}{\lambda - \mu} \\ &\to \frac{A}{\gamma}\lambda^{\alpha} \mp i\pi u(\lambda) \text{ as } \lambda \to \text{ the interval, } -2a \leq \lambda \leq 2a. \end{aligned} \tag{6.10}$$

We see directly that $F(\lambda)$ has the following properties,

(i) It is analytic in the cut complex λ-plane, $(-2a \leq \lambda \leq 2a)$.

(ii) It behaves like $1/\lambda$ as $\lambda \to \infty$ by (6.8).

(iii) It is real for λ real and $|\lambda| > 2a$.

These conditions suffice to insure a unique function F. This uniqueness is a consequence of the fact that the real part of $F(\lambda)$ is uniquely determined. It is a standard result of potential theory that it is unique because it satisfies Laplace's equation with the boundary conditions (6.10) in the region of analyticity given by (i) above and vanishes at infinity by (ii). The imaginary part is then determined, up to a constant, from the real part by the solution of the Cauchy-Riemann equations. That constant is determined by condition (iii) above, which completes the assurance of uniqueness.

In the special case that α is an odd integer the solution can be given in closed form. Let us try

$$F(\lambda) = \frac{A}{\gamma}\left(\lambda^\alpha - \left\{\lambda^{\alpha-1}\left[1 - \frac{4a^2}{\lambda^2}\right]^{-\frac{1}{2}}\right\}\sqrt{\lambda^2 - 4a^2}\right), \qquad (6.11)$$

where $\{\,\}$ denotes the polynomial part in λ when the quantity inside has been expanded in a power series in inverse powers of λ. The value of a is determined by equating the coefficient of λ^{-1} to γ as required by (6.8-9). These equations are straightforward to derive and they are,

$$(\alpha = 1) \quad 2Aa^2 = \gamma, \quad (\alpha = 3) \quad 6Aa^4 = \gamma, \quad (\alpha = 5) \quad 20Aa^6 = \gamma,$$
$$\cdots, \binom{-\frac{1}{2}}{\frac{1}{2}(\alpha+1)} A(-4a^2)^{\frac{1}{2}(\alpha+1)} = \gamma, \cdots . \qquad (6.12)$$

The function F so determined satisfies all the required conditions and so gives the solution to (6.9). It is, by comparison of (6.11) with (6.10),

$$u(\lambda) = \frac{A}{\pi\gamma}\left\{\lambda^{\alpha-1}\left[1 - \frac{4a^2}{\lambda^2}\right]^{-\frac{1}{2}}\right\}\sqrt{4a^2 - \lambda^2}, \quad -2a \le \lambda \le 2a \quad (6.13)$$

where a is given by (6.12) and as above $\{\,\}$ denotes the polynomial part.

The value of the integrand at the saddle-point, which gives the leading order in the asymptotic behavior of the Toeplitz determinant, is

$$\exp\left[\gamma \sum_{1 \le i < j}^{N} \ln|\lambda_i - \lambda_j| - \sum_{i=1}^{N} V(\lambda_i)\right], \qquad (6.14)$$

which in the limit as $N \to \infty$ becomes,

$$\asymp \exp\left[N^2\Big(\frac{\gamma \ln N}{2(\alpha+1)} + \frac{\gamma}{2}\int_0^1\int_0^1 \ln|\lambda(x)-\lambda(y)|\,dx\,dy \right.$$
$$\left. - \frac{A}{\alpha+1}\int_0^1 (\lambda(x))^{\alpha+1}\ dx\Big)\right],$$
$$\asymp \exp\left[N^2\Big(\frac{\gamma \ln N}{2(\alpha+1)} + \frac{\gamma}{2}\int_{-2a}^{2a}\int_{-2a}^{2a} \ln|\lambda-\mu| u(\lambda)u(\mu)\,d\lambda\,d\mu \right.$$
$$\left. - \frac{A}{\alpha+1}\int_{-2a}^{2a} \lambda^{\alpha+1} u(\lambda)\,d\lambda\Big)\right]. \tag{6.15}$$

This equation can be reduced by integrating the saddle-point equation (6.9) with respect to λ from 0 to λ, and by use of the normalization condition (6.8) to give,

$$\bar{Z}_g(N) \asymp N! \exp\left[\tfrac{1}{2}N^2\left\{\frac{\gamma \ln N}{\alpha+1}\right.\right.$$
$$\left.\left. + \int_{-2a}^{2a}\left(\gamma \ln|\lambda| - \frac{A}{\alpha+1}\lambda^{\alpha+1}\right) u(\lambda)\,d\lambda\right\}\right], \tag{6.16}$$

to leading order in N. The factor of $N!$ comes from the $N!$ identically valued saddle-points due to permutations of the λ_i. The result (6.16) for the asymptotic behavior of the Toeplitz determinant ($\gamma = 2$) and its generalizations for other exponents of Δ is also valid for all values of $\alpha > -1$, $A > 0$ and not just when α an odd integer. In the cases where α is not an odd integer, we have not given an explicit solution for $u(\lambda)$, but as remarked above it is determined by (6.9) and the conditions (i)-(iii). These results represent an extension of the work of Szegő [14] to a class of measures with infinite instead of finite support, and following the analysis of section 2, also provide the asymptotic behavior of the normalization constants for the orthogonal polynomials with respect to these measures. The case $\alpha = 1$, $\gamma = 2$ discussed above corresponds to the results (2.10) for the Hermite polynomials, and the leading behavior, $\exp[\frac{1}{2}N^2 \ln N]$ is correctly given. The next order behavior is of the order $\exp[KN^2]$ for a constant K and it also can be obtained from (6.16) and agrees with that derived from (2.10).

VII. Asymptotic Behavior for Measures of the Quantum Gravity Type

In the problem we introduced in section 1 the measure had the property (1.6) that it depended on the order of the determinant being considered. In this case, the analysis of the previous section is slightly modified. Specifically, we need to treat potentials of the structure,

$$V(\lambda) = Nv\left(\frac{\lambda}{\sqrt{N}}\right), \tag{7.1}$$

where the N dependence is now explicit. If we now investigate the asymptotic behavior of (6.1), again following the method of Brézin *et al.*, [4] but now with the potential of the structure (7.1) instead of (6.3), we make the choice $\beta = \frac{1}{2}$ instead of (6.5) and obtain,

$$v'(\lambda(x)) = \gamma \fint_0^1 \frac{dy}{\lambda(x) - \lambda(y)}, \tag{7.2}$$

instead of (6.6). By means of exactly parallel analysis to that of the previous section, we derive the equation,

$$v'(\lambda) = \gamma \fint_{-2a}^{2a} \frac{u(\mu)\, d\mu}{\lambda - \mu}, \text{ for } -2a \leq \lambda \leq 2a, \tag{7.3}$$

in place of (6.9) but still subject to the conditions (i)-(iii) of section 6. The asymptotic behavior is now given in terms of the $u(\lambda)$ of (7.1) by

$$\bar{Z}_g(N) \asymp N! \exp\left[\tfrac{1}{2}N^2 \left\{\frac{1}{2}\gamma \ln N + \int_{-2a}^{2a} (\gamma \ln|\lambda| - v(\lambda))\, u(\lambda)\, d\lambda\right\}\right], \tag{7.4}$$

to leading order in N.

For the special case,

$$v(\lambda) = \tfrac{1}{2}\lambda^2 + g\lambda^4, \tag{7.5}$$

the methods are known [5, 12] to compute the leading terms in the asymptotic expansion in powers of $1/N$ of the exact solution of the quantity,

$$-\frac{1}{N^2} \ln\left(\frac{\bar{Z}_N(g)}{\bar{Z}_N(0)}\right) \asymp E_0^{(\gamma)}(g) + \frac{1}{N}E_1^{(\gamma)}(g) + \frac{1}{N^2}E_2^{(\gamma)}(g) + \cdots, \tag{7.6}$$

for the values $\gamma = 1, 2, 4$, a number of the leading terms are known explicitly. They are, for $\gamma = 2$,

$$
\begin{aligned}
E_0^{(2)}(g) &= \tfrac{1}{24}(f_2 - 1)(9 - f_2) - \tfrac{1}{2}\ln f_2, \\
E_1^{(2)}(g) &= 0, \\
E_2^{(2)}(g) &= 3g + \tfrac{1}{12}\left[\ln f_2 - 12g\right] - 96g^2 \int_0^1 \frac{(1-x)\,dx}{(1+48gx)^2},
\end{aligned}
\tag{7.7}
$$

with

$$
f_2(g) = \frac{\sqrt{1+48g} - 1}{24g}. \tag{7.8}
$$

For $\gamma = 1$ the series terms are,

$$
\begin{aligned}
E_0^{(1)}(g) &= \tfrac{1}{48}(f_1 - 1)(9 - f_1) - \tfrac{1}{4}\ln f_1, \\
E_1^{(1)}(g) &= \tfrac{1}{4}\ln f_1 + \tfrac{1}{8}(1 - f_1) - \tfrac{1}{2}\int_0^1 \ln F(x, \tfrac{1}{4}g)\,dx,
\end{aligned}
\tag{7.9}
$$

with

$$
f_1(g) = \frac{\sqrt{1+24g} - 1}{12g}, \tag{7.10}
$$

and

$$
F(x, g) = \frac{x + 8g\rho^2 - \sqrt{(x + 24g\rho^2)(x - 8g\rho^2)}}{32g^2\rho^3}, \tag{7.11}
$$

where

$$
\rho = \frac{\sqrt{1+96gx} - 1}{48g}. \tag{7.12}
$$

Lastly, for the case $\gamma = 4$, the results are,

$$
\begin{aligned}
E_0^{(4)}(g) &= \tfrac{1}{12}(f_4 - 1)(9 - f_4) - \ln f_4, \\
E_1^{(4)}(g) &= -\tfrac{1}{2}\ln f_4 - \tfrac{1}{4}(1 - f_4) + \int_0^1 \ln F(x, g)\,dx,
\end{aligned}
\tag{7.13}
$$

with

$$f_4(g) = \frac{\sqrt{1+96g}-1}{48g}, \tag{7.14}$$

and $F(x,g)$ as given by (7.11).

There are some interesting relations between these results that become more evident in terms of the following inequality which we derive. For this purpose it is useful to write $V(\lambda)$ of (1.6) as

$$V(\lambda) = \tfrac{1}{2}\lambda^2 + gW(\lambda), \tag{7.15}$$

where as we will work at fixed N in what follows, no explicit assumption is made now about the N dependence of the coefficients of W, however we will considered that it is asymptotically positive and tends to infinity as $\lambda \to \pm\infty$. We will use the notation $k^{-1} + k'^{-1} = 1$, with necessarily $k > 1$. Thus we may write (6.1) as

$$\begin{aligned}
\bar{Z}_N^{(\gamma)}(g) &= \int_{-\infty}^{\infty}\!\!\cdots\int \Delta^{\gamma}(\lambda)\prod_{i=1}^{N}\exp\left[-\frac{\lambda_i^2}{2k} - gW(\lambda_i)\right]\exp\left[-\frac{\lambda_i^2}{2k'}\right]d\lambda_i, \\
&\le \left\{\int_{-\infty}^{\infty}\!\!\cdots\int \Delta^{\gamma k}(\lambda)\prod_{i=1}^{N}\exp\left[-\frac{\lambda_i^2}{2} - kgW(\lambda_i)\right]d\lambda_i\right\}^{1/k} \\
&\quad\times\left\{\int_{-\infty}^{\infty}\!\!\cdots\int \prod_{i=1}^{N}\exp\left[-\frac{\lambda_i^2}{2}\right]d\lambda_i\right\}^{1/k'} \\
&= (2\pi)^{\frac{k-1}{2k}N}\left[\bar{Z}_N^{(\gamma k)}(kg)\right]^{1/k},
\end{aligned} \tag{7.16}$$

where the inequality follows from the Hölder inequality for integrals. Alternately we can write the result (7.16) as

$$\bar{Z}_N^{(\gamma k)}(g) \ge (2\pi)^{-\frac{(k-1)N}{2}}\left[\bar{Z}_N^{(\gamma)}(g/k)\right]^{k}, \tag{7.17a}$$

$$\bar{Z}_N^{(\gamma/k)}(g) \le (2\pi)^{\frac{(k-1)N}{2k}}\left[\bar{Z}_N^{(\gamma)}(kg)\right]^{1/k}, \tag{7.17b}$$

which gives a lower bound (accurate of course to all orders in $1/N$) for any larger value of γ in terms of a reference case, and an upper bound for any smaller value of γ. If the bound (7.17) were exactly true then the coefficients $E_j^{(\gamma)}(g)$ would have the property,

$$E_j^{(\gamma k)}(g) = kE_j^{(\gamma)}(g/k). \tag{7.18}$$

If we compare with the exactly known results, since $u(\lambda)$ depends only on A/γ or g/γ respectively, we see to that (6.16) and (7.4) have this property. Hence this inequality has the property that it is exactly satisfied as an equality to the same leading order in $1/N$ for which (6.16) and (7.4) are valid. We find that this property is good through the first two orders in $1/N$, namely the orders $N^2 \ln N$ and N^2. Comparison with the exact solutions (7.7-14) shows that this property is valid for $E_0^{(\gamma)}(g)$ but fails for $E_1^{(\gamma)}(g)$, that is, at order N which is the next order. Nevertheless the inequalities (7.17) applied with $\gamma = 2$ as the reference case do correctly indicate the opposite signs found in (7.9) and (7.13) for $E_1^{(1)}$ and $E_1^{(4)}$. Thus we conclude that, in line with the arguments in section 6 and the beginning of this section, that as the validity of the saddle-point method is sufficient to insure that our inequalities are equalities, because then the neighborhood of only one point (or here $N!$ symmetrical points) is all that contributes to the integral, and the inequalities are constructed so that the value of that integrand is preserved, we see that even in cases where we do not have the explicit solution, we still have the relationship between the results for different values of γ given by the rigorous inequalities, and that they are saturated at the first two orders in $1/N$.

References

1. J. Ambjørn, B. Durhuus and J. Fröhlich, *Diseases of Triangulated Random Surface Models and Possible Cures*, Nucl. Phys. **B257[FS14]** (1985), 433–449.
2. F. David, *Planar diagrams, two dimensional lattice gravity and surface models*, Nucl. Phys. **B257[FS14]** (1985), 45–58.
3. V. A,. Kazakov, I. K. Kostov and A. A. Migdal, *Critical Properties of Randomly Triangulated Planar Random Surfaces*, Phys. Letts. **157B** (1985), 295–300.
4. E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Planar Diagrams*, Commun. Math. Phys. **59** (1978), 35–51.
5. D. Bessis, C. Itzykson and J.-B. Zuber, *Quantum Field Techniques in Graphical Enumeration*, Adv. Appl. Math. **1** (1980), 109–157.
6. M. Douglas and S. Shenker, *Strings in less than one dimension*, Nucl. Phys. **B335** (1990), 635–654.
7. E. Brézin and V. Kazakov, *Exactly Solvable Field Theories of Closed Strings*, Phys. Letts. **B236** (1990), 144–150.
8. D. J. Gross and A. A. Migdal, *Nonperturbative Two-Dimensional Quantum Gravity*, Phys. Rev. Letts. **64** (1990), 127-130.
9. J. Zinn-Justin, *2D Gravity and Random Matrices,*, 1990 Saclay preprint SPhT 90-065..
10. M. L. Mehta, *Random Matrices*, (second edition), Academic Press, New York, 1990.
11. J. D. Bessis, *A new method for the combinatorics of the topological expansion*, Commum. Math. Phys. **69** (1979), 147–163.

12. M. L. Mehta and G. Mahoux, *A Method of Integration over Matrix Variables: III,*, 1990 Saclay preprint..
13. B. Nienhuis, *Coulomb Gas Formulation of Two-dimensional Phase Transitions*, Phase Transitions and Critical Phenomena, Vol. 11, C. Domb and J. L. Lebowitz (eds.), vol. 11, Academic Press, London, 1987.
14. G. Szegő, *Orthogonal Polynomials*, Colloquium Publications, no. 23, American Mathematical Society, New York, 1939, U. Grenander and G. Szegő, "Toeplitz Forms and Their Applications," University of California Press, Berkeley, 1958..
15. G. A. Baker, Jr., *Essentials of Padé Approximants*, Academic Press, New York, 1975.
16. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Parts I and II, McGraw-Hill, New York, 1953.
17. L. Habsieger, *Conjectures de MacDonald et q-Intégrale de Selberg-Askey*, (Thèse), l'Institut de Recherche Mathématique Avancée, Strasbourg, 1987.
18. A. Selberg, *Bemerkninger on et multipelt integral*, Norsk. Mat. Tidsskr. **26** (1944), 71-78.
19. G. A. Baker, Jr. and P. R. Graves-Morris, *Padé Approximants Part I: Basic Theory*, Encyclopedia of Mathematics and its Applications, Vol. 13, G.-C. Rota (ed.), Cambridge University Press, London, 1981.
20. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, London, 1927.
21. L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, London, 1966.
22. G. Gasper and M. Rahman, *Basic Hypergeometric Functions*, Encyclopedia of Mathematics and its Applications, Vol. 35, G.-C. Rota (ed.), Cambridge University Press, London, 1990.
23. E. Heine, J. Reine Angew. Math. **34** (1845), 285–328.
24. W. B. Jones and W. J. Thron, *Continued Fractions, Analytic Theory and Applications*, Encyclopedia of Mathematics and its Applications, Vol. 11, G.-C. Rota (ed.), Cambridge University Press, London, 1980.
25. L. C. Biedenharn and M. A. Lohe, *Quantum Groups and Basic Hypergeometric Functions*, Duke University preprint, 1990..
26. E. Picard, *Leçons sur Quelques Equations Fonctionnelles*, Gauthier Villars, Paris, 1928.
27. M. E. Kuczma, *Functional Equations in a Single Variable*, Monografie Mat. **46**, Polish Scientific Publishers, Warsaw, 1968.